Announcements

1) HW #3 up Thursday,
due next week

Example 1: (heat equation)

Solve

$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(x,t),$$

where
$$\frac{\partial u}{\partial x}(0,t) = 0$$

$$\frac{\partial u}{\partial x}(\pi,t) = 0$$

$$u(x,0) = 1$$

We started by wishing

that

$$u(x,t) = f(x)g(t),$$

we obtained

$$\frac{1}{2}\frac{f''(x)}{f(x)} = \frac{g'(t)}{g(t)}, \text{ so}$$

$$\frac{f''(x)}{f(x)} = 2\frac{g'(t)}{g(t)} = \alpha$$

We can solve

$$2\frac{g'(t)}{g(t)} = \alpha$$ by integration:

$$2\ln g(t) = \alpha t + D_1,$$

$$g(t) = e^{\frac{\alpha t + D_1}{2}}.$$

If $f(x) = e^{rx}$, we

get

$f''(x) - \alpha f(x) = 0,$

$r^2 e^{rx} - \alpha e^{rx} = 0,$

So $r^2 = \alpha$.

Suppose $\boxed{\alpha < 0}$

Then

$$r = \pm i \underbrace{\sqrt{-\alpha}}_{s}$$

Solution to

$$y''(x) - \alpha y(x) = 0$$

look like

$$\boxed{c_1 e^{isx} + c_2 e^{-isx} = y(x)}$$

Initial condition

$$\frac{\partial u}{\partial x}(0,t) = 0.$$

If $u(x,t) = f(x)g(t)$,

$$\frac{\partial u}{\partial x} = f'(x)g(t)$$

$$0 = \frac{\partial u}{\partial x}(0,t) = f'(0)g(t)$$

Since $u(x,0) = 1$, $g(t) \neq 0$.

This says $f'(0)=0$,

so if

$$f(x) = c_1 e^{isx} + c_2 e^{-isx},$$

and

$$f'(x) = isc_1 e^{isx} - isc_2 e^{-isx},$$

$$0 = f'(0) = is(c_1 - c_2),$$

so $c_1 - c_2 = 0$ and

$$c_1 = c_2.$$

If $C_1 = C_2$, then

$$f(x) = C_1\left(e^{isx} + e^{-isx}\right)$$

$$= C_1\left(2 \cdot \cos(sx)\right)$$

since we showed last time that

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

Now use the remaining initial condition,

$$\frac{\partial u}{\partial x}(\pi, t) = 0$$

Remember $\frac{\partial u}{\partial x} = f'(x)g(t)$,

so substituting,

if $f(x) = c_1(2\cos(sx))$,

$$f'(x) = (-2c_1 s)\sin(sx)$$

Then

$$\frac{\partial u}{\partial x} = (-2C_1 s)(\sin(sx))g(t),$$

$$0 = \frac{\partial u}{\partial x}(\pi, t)$$
$$= (-2C_1 S)\sin(\pi S)\, g(t).$$

Since $u(x,0) = 1$, $C_1 \neq 0 \neq g(t)$,

so $\sin(\pi s) = 0$.

This says S has to be of the form

$S = n$ for n an integer ($n = 0, \pm 1, \pm 2, \ldots$).

We then have

$$\boxed{u(x,t) = 2C_1 \cos(nx)\, e^{\frac{\alpha t + D_1}{2}}}$$

where $\alpha = -S^2$ and n is an integer.

For more explicit solutions, take Fourier Series (Math 454).

Nonhomogeneous Equations

(section 4.4)

We know how to solve

$$y'' + 3y' + 2y = 0,$$ what

about

$$y'' + 3y' + 2y = t\,?$$

Guess and check

What if $y(t) = mt + b$?

Then $y'(t) = m$, $y''(t) = 0$.

The equation becomes

$$0 + 3m + 2(mt + b) = t.$$

Then $(2m - 1)t + (3m + 2b) = 0$

Then

$$2m - 1 = 0, \text{ so } m = \frac{1}{2}$$

and

$3m + 2b = 0$, so

$$2b = -\frac{3}{2},$$

$b = -3/4$. Then

$$\boxed{y = \frac{t}{2} - \frac{3}{4}} \text{ is}$$

a solution to our equation.

Now if y, z are solutions to $y'' + 3y' + 2y = t$,

then

$$y'' + 3y' + 2y = t$$

$$z'' + 3z' + 2z = t,$$

and subtracting,

$$y'' - z'' + 3(y' - z') + 2(y - z) = 0.$$

Using linearity of the derivative,

$$(y-z)'' + 3(y-z)' + 2(y-z) = 0,$$

So $y-z$ is a solution to the homogeneous equation

$$f'' + 3f' + 2f = 0.$$

All solutions to the homogeneous equation are of the form

$$f(t) = c_1 e^{-t} + c_2 e^{-2t},$$

so

$$y - z = c_1 e^{-t} + c_2 e^{-2t}, \text{ so}$$

$$z = y - c_1 e^{-t} - c_2 e^{-2t}$$

$$= \boxed{\left(\frac{t}{2} - \frac{3}{4}\right) - c_1 e^{-t} - c_2 e^{-2t}}$$

Observation: (two solutions)

Any two solutions to

$$ay'' + by' + cy = f(t)$$

where a, b, c are constants

differ by a solution of

$$ay'' + by' + cy = 0.$$

So we only need one solution

of the nonhomogeneous equation!

General Method: Variation of Parameters (Section 4.6)

Another outrageous trick

Any solution to

$$ay'' + by' + cy = 0$$

is a linear combination of two linearly independent functions: $c_1 f(t) + c_2 g(t)$.

What if, for nonhomogeneous equations, we replace the constants c_1 and c_2 with nonconstant functions of t?